Announcements

1) HW 1 Supplement due tomorrow
2) HW 2 up due next week
3) Ford Day –
   Wednesday
   12:45 - 2 Kochoff
   (registration @ 12)
   3:45 - 5 COB Michigan
   Room East

Back to Example:

$$\frac{dy}{dx} = xy^2 - x + y^2 - 1$$

if $y(4) = 1$.

We factored

$$xy^2 - x + y^2 - 1 = x(y^2-1) + (y^2-1)$$
$$= (x+1)(y^2-1)$$

Rewrite

$$\frac{dy}{dx} = (x+1)(y^2-1),$$

divide both sides by y^2-1 to get

$$\frac{1}{y^2-1}\frac{dy}{dx} = x+1$$

Ignore $\frac{dy}{dx}$, integrate left hand side wrt y, right hand side wrt x.

$$\int \frac{1}{y^2-1}\,dy = \int (x+1)\,dx$$

$$= \frac{x^2}{2} + x + C$$

$\int \frac{1}{y^2-1}\,dy$ use partial fractions!

$$\frac{1}{y^2-1} = \frac{1}{(y-1)(y+1)}$$

$$= \frac{1/2}{y-1} + \frac{-1/2}{y+1}$$

$$\int \frac{1}{y^2-1}\,dy$$

$$= \frac{1}{2}\int\left(\frac{1}{y-1} - \frac{1}{y+1}\right)dy$$

$$= \frac{1}{2}\left(\ln|y-1| - \ln|y+1|\right)$$

$$= \frac{1}{2}\ln\frac{|y-1|}{|y+1|}$$

$$= \frac{x^2}{2} + x + C$$

Solve for C using $y(4)=3$

$$\frac{1}{2}\ln\left(\frac{2}{4}\right) = 8+4+C$$

$$C = \frac{1}{2}\ln(1/2) - 12$$

$$= \frac{1}{2}(\ln(1) - \ln(2)) - 12$$

$$= -\frac{\ln(2)}{2} - 12$$

$$= -\ln\sqrt{2} - 12$$

Now see if we can solve for y! Leave C in.

$$\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = \frac{x^2}{2} + x + C$$

$$\ln\left|\frac{y-1}{y+1}\right| = x^2 + 2x + 2C$$

Exponentiate:

$$\left|\frac{y-1}{y+1}\right| = e^{x^2+2x+2C}$$

Assume $y \geq 1$. Then

$$\left|\frac{y-1}{y+1}\right| = \frac{y-1}{y+1}$$

$$\frac{y-1}{y+1} = e^{x^2+2x+2C}$$

$$(y-1) = (y+1)e^{x^2+2x+2C}$$

$$y - ye^{x^2+2x+2c} = e^{x^2+2x+2c} + 1$$

$$y\left(1 - e^{x^2+2x+2c}\right) = e^{x^2+2x+2c} + 1$$

so

$$y = \left(1 + e^{x^2+2x+2c}\right) \Big/ \left(1 - e^{x^2+2x+2C}\right)$$

Since $C = -\ln\sqrt{2} - 12$,

$$\boxed{y = \frac{1 + e^{x^2 + 2x - 2(\ln\sqrt{2} - 12)}}{1 - e^{x^2 + 2x - 2(\ln\sqrt{2} - 12)}}}$$

Example 1: (The brine problem)

Suppose a brine containing .3 kg of salt per liter runs into a tank initially filled with 400 L of water containing 2 kg of salt. If the brine enters at 10 L/min, the mixture is kept uniform by stirring, and $\longrightarrow$

/

the mixture flows out at the same rate, how much salt (in kg) is in the tank after 10 min?

Label amount of salt left in tank at time t by $S(t)$. Since the tank starts with 2 kg of salt in it, $S(0)=2$.

Make a differential equation:

$$\frac{ds}{dt} = (\text{rate in}) - (\text{rate out})$$

$$\text{rate in} = (.3\ \text{kg}/\text{L})(10\ \text{L}/\text{min})$$
$$= 3\ \text{kg}/\text{min}$$

since mixture flows in at 10 L/min, has .3 kg/L of salt.

$$\text{rate out} = \left(\frac{S(t)\,\text{kg}}{400\,\text{L}}\right)\cdot\left(10\ \text{L/min}\right)$$

$$= \frac{S(t)}{40}\ \text{kg/min}$$

Since mixture exits at 10 L/min, and there are $S(t)/400$ kg/L of salt.

Differential equation:

$$\frac{dS}{dt} = 3\ \text{kg/min} - \frac{S(t)}{40}\ \text{kg/min}$$

Solve! Divide by

$$3 - \frac{s(t)}{40}.$$

$$\frac{1}{3 - \frac{s(t)}{40}} \frac{ds}{dt} = 1$$

$$\frac{1}{3 - s/40} = 1$$

Integrate rhs wrt t,

lhs wrt s.

$$\int \frac{1}{3 - s/40}\, ds = t + C$$

$$-40 \ln \left| 3 - s/40 \right| = t + C$$

Solve for C using $s(0) = 2$.

$$-40 \ln \left| 3 - 1/20 \right| = C$$

$$C = -40 \ln \left(\frac{59}{20} \right).$$

Solve for S:

$$\ln|3 - S/40| = \frac{(t+c)}{-40}$$

$$|3 - S/40| = e^{\frac{-t-c}{40}}$$

$$3 - S/40 = e^{-t/40} e^{-c/40}$$

$$S = 120 - 40e^{-t/40} e^{-c/40}$$

Since $C = -40 \ln(59/20)$

$$\boxed{S = 120 - 118e^{-t/40}}$$

Final answer: